PARTICULAR SOLUTIONS OF EINSTEIN'S RECENT UNIFIED THEORIES*

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ABSTRACT Different field equations, one proposed by Einstein and Straus in 1946 and the other by Einstein in 1950, have been solved for the particular case analogous to an infinite charged plane.

INTRODUCTION

Einstein has developed a new generalized theory (Einstein and Straus, 1946) in which the field variables are the sixteen components of a non-symmetric tensor $g_{\mu\nu}(\mu, \nu = 1, 2, 3, 4)$. It has been suggested by Einstein that the antisymmetric part of this tensor corresponds to the electromagnetic field.

In order to obtain the field equations one has first to obtain the non-symmetric Γ's from the sixty-four equations

$$g_{ik,a} - g_{sk}\Gamma^s_{ia} - g_{is}\Gamma^s_{ik} = 0 \qquad \ldots (1)$$

and then obtain P's defined by

$$P_{ik} = \Gamma^a_{ik,a} - \tfrac{1}{2}(\Gamma^a_{ia,k} + \Gamma^a_{ak,i}) - \Gamma^a_{ib}\Gamma^b_{ak} + \tfrac{1}{2}\Gamma^a_{ik}(\Gamma^b_{ab} + \Gamma^b_{ba}) \qquad \ldots (2)$$

and finally write down the equations

$$\tfrac{1}{2}(\Gamma^a_{ia} - \Gamma^a_{ai}) = 0 \qquad \ldots (3a)$$

$$\tfrac{1}{2}(P_{ik} + P_{ki}) = 0 \qquad \ldots (3b)$$

$$\tfrac{1}{2}(P_{ik,l} - P_{ki,l}) + \tfrac{1}{2}(P_{kl,i} - P_{lk,i}) + \tfrac{1}{2}(P_{li,k} - P_{il,k}) = 0 \qquad \ldots (3c)$$

(in terms of g's) which comprise the field equations propounded in 1946. It may be mentioned in passing that there exists no simple method of solving equation (1). Straus (1949), however, has recently developed a method which reduces the labour of calculations to some extent.

Einstein (1950), however, has very recently modified the field equations The new equations are equivalent to equations (3*a*), (3*b*) and the equation

$$\tfrac{1}{2}(P_{ik} - P_{ki}) = 0 \qquad \ldots (3d)$$

instead of (3*c*). We shall designate the field equations given by (3*a*), (3*b*), (3*c*) as I and those given by (3*a*), (3*b*), (3*d*) as II.

It may be noted that II is stronger than I. This paper dicusses a particular solution of the above field equations.

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CALCULATIONS

We take

$$g_{11}=1,\ g_{22}=g_{33}=G(x_1),\qquad g_{44}=H(x_1),\qquad g_{14}=-g_{41}=I(x_1)$$

The sixty-four equations (1) now break up into a number of partial sets and we obtain the following non-vanishing Γ's :

$$\Gamma^1_{44}=-\tfrac{1}{2}\frac{dH}{dx}-2I\frac{HI'-\frac{1}{2}IH'}{I^2+H}$$

$$\Gamma^1_{14}=-\Gamma^1_{41}=\frac{HI'-\frac{1}{2}IH'}{I^2+H}$$

$$\Gamma^4_{14}=\Gamma^4_{41}=\frac{II'+\frac{1}{2}H'}{I^2+H}$$

$$\Gamma^2_{12}=\Gamma^2_{21}=\Gamma^3_{13}=\Gamma^3_{31}=\frac{1}{2G}\cdot\frac{dG}{dx}.$$

$$\Gamma^1_{22}=\Gamma^1_{33}=-\tfrac{1}{2}\frac{dG}{dx}$$

$$\Gamma^2_{42}=-\Gamma^2_{24}=\Gamma^3_{43}=-\Gamma^3_{34}=-\frac{I}{2G}\cdot\frac{dG}{dx},$$

dash denoting differentiation with respect $x(x=x_1)$.

The expressions for P's are now calculated by formula (2) as follows :

$$P_{11}=-\frac{d}{dx}\left(\frac{G'}{G}+\frac{II'+\frac{1}{2}H'}{I^2+H}\right)-2\left(\frac{G'}{2G}\right)^2-\left(\frac{II'+\frac{1}{2}H'}{I^2+H}\right)^2$$

$$P_{14}=\tfrac{1}{2}\frac{d}{dx}\left(\frac{HI'-\frac{1}{2}IH'}{I^2+H}\right)-\frac{d}{dx}\left(\frac{IG'}{2G}\right)-2.\ \frac{G'}{2G}\cdot\frac{IG'}{2G}+2.\frac{G'}{2G}\left(\frac{HI'-\frac{1}{2}IH'}{I^2+H}\right)$$

$$P_{22}=P_{33}=-\frac{d}{dx}\left(-\tfrac{1}{2}G'\right)-\tfrac{1}{2}G'.\frac{II'+\frac{1}{2}H'}{I^2+H}$$

$$P_{44}=\frac{d}{dx}\left[-\tfrac{1}{2}\frac{dH}{dx}-2I\frac{HI'-\frac{1}{2}IH'}{I^2+H}\right]+\left(\frac{HI'-\frac{1}{2}IH'}{I^2+H}\right)^2-2\left(\frac{IG'}{2G}\right)^2$$
$$+\left(-\tfrac{1}{2}\frac{dH}{dx}-2I\frac{HI'-\frac{1}{2}IH'}{I^2+H}\right)\left(2\frac{G'}{2G}-\frac{II'+\frac{1}{2}H'}{I^2+H}\right)$$

other P's vanish.

We now proceed to form field equations I. Of the four equations (3a) three are satisfied identically and we are left with only one equation; *viz.*,

$$\frac{HI'-\frac{1}{2}IH'}{I^2+H}+\frac{IG'}{G}=0 \qquad\qquad (4a)$$

Of the ten equations (3b) five are identically satisfied due to vanishing of corresponding P's, one is identically satisfied due to antisymmetry of P_{14}, and of the rest two become identical. So we get the three equations

$$\frac{d}{dx}\left(\frac{G'}{G}+\frac{II'+\frac{1}{2}H'}{I^2+H}\right)+\frac{1}{2}\left(\frac{G'}{G}\right)^2+\left(\frac{II'+\frac{1}{2}H'}{I^2+H}\right)^2=0 \qquad \ldots \quad (4b)$$

$$G''+G'.\frac{II'+\frac{1}{2}H'}{I^2+H}=0 \qquad \ldots \quad (4c)$$

$$\frac{d}{dx}\left(\frac{1}{2}H'+\frac{HI'-\frac{1}{2}IH'}{I^2+H}\right)-\left(\frac{HI'-\frac{1}{2}IH'}{I^2+H}\right)^2+\frac{1}{2}\left(\frac{IG'}{G}\right)^2$$
$$+\left(\frac{1}{2}H'+\frac{HI'-\frac{1}{2}IH'}{I^2+H}\right)\left(\frac{G'}{G}-\frac{II'+\frac{1}{2}H'}{I^2+H}\right)=0 \qquad \ldots \quad (4d)$$

Equation ($3c$) is identically satisfied because P_{14} is antisymmetric and other non-diagonal P's vanish.

It may be noted that the field equations I are not affected by the expression for P_{14}.

In field equations II, the contributions of ($3a$), ($3b$) are the same, but ($3d$) give an additional equation corresponding to

$$\tfrac{1}{2}(P_{14}-P_{41})=0$$

which leads to the additional equation

$$\frac{1}{2}\frac{d}{dx}\left(\frac{HI'-\frac{1}{2}IH'}{I^2+H}\right)-\frac{d}{dx}\left(\frac{IG'}{2G}\right)-\frac{IG'^2}{2G^2}+\frac{G'}{G}\left(\frac{HI'-\frac{1}{2}IH'}{I^2+H}\right)=0 \qquad \ldots \quad (4e)$$

SOLUTION OF FIELD EQUATIONS I

We are to solve for the three quantities, G, H, I, from the four equations ($4a$), ($4b$), ($4c$), ($4d$).

From ($4c$) we get

$$I^2+H=\frac{a}{G'^2}, \ (a, \text{ constant}) \qquad \ldots \quad (5a)$$

Substituting from ($4c$) in ($4b$) we have

$$\frac{d}{dx}\left(\frac{G'}{G}-\frac{G''}{G'}\right)+\frac{1}{2}\left(\frac{G'}{G}\right)^2+\left(\frac{G''}{G'}\right)^2=0$$

From this we get, after some manipulation

$$G'^2=b.G^{\frac{1}{2}}e^{cG^2}, \ (b,\ c, \text{ constants}), \qquad \ldots \quad (5b)$$

($4a$) can be written as

$$\frac{(H+I^2)I'-I(II'+\frac{1}{2}H')}{H+I^2}+\frac{IG'}{G}=0,$$

which leads to the solution

$$\frac{IG}{\sqrt{I^2+H}}=\text{const},$$

which again, on using (5a), reduces to

$$I=\frac{d}{GG'}\,,\ (d,\ \text{constant}) \qquad \dots \quad 5c)$$

(5a), (5b), (5c), obtained without using (4d), determine G, H, I completely; because G being known by (5b), I can be determined from (5c) and can then be used to determine H from (5a). We proceed to obtain an expression using (4d) in order to test the consistency of (5a), (5b), (5c) with (4d).

Using (4a) and (5c), (4d) can be written as

$$\frac{d\lambda}{dx}+\frac{d}{dx}(\log GG').\ \lambda=\frac{3}{2}\frac{d^2}{G^4}$$

where
$$\lambda=\tfrac{1}{2}H'+2I\frac{HI'-\frac{1}{2}IH'}{I^2+H}$$

the solution of which, on using (4a) again, can be written as

$$H'=\frac{1}{GG'}\left(\frac{5}{2}.\frac{d^2}{G^2}+e\right),\ (e,\ \text{constant})$$

We shall test the consistency of this with (5a), (5b) and (5c)
(5a) and (5c) give

$$H=\frac{1}{G'^2}\left(a-\frac{d^2}{G^2}\right)$$

Differentiating and using

$$\frac{2G''}{G'^2}=\frac{1}{2G}+2cG$$

which can easily be derived from (5b), we have

$$H'=\frac{1}{GG'}\left(\frac{5}{2}.\frac{d^2}{G^2}+2acG^2+2cd^2-\tfrac{1}{2}a\right) \qquad \dots \quad (5e)$$

Consistency of (5d) and (5e) requires

$$c=0,\ e=-\frac{1}{2}a$$

The full solutions of g's with these restrictions on the constant lead to

$$g_{11}=1\ \ g_{22}=g_{33}=\left(k+\tfrac{3}{4}\sqrt{b}.x\right)^{4/3},g_{44}=\frac{16}{9(k+\frac{3}{4}\sqrt{b}x)^{2/3}}\left(a-\frac{d^2}{(k+\frac{3}{4}\sqrt{b}x)^{8/3}}\right)$$

$$g_{14}=-g_{41}=\frac{4d}{3\sqrt{b}(k+\frac{3}{4}\sqrt{b}x)^{5/3}}$$

If g_{14}, and therefore I, be interpreted as the electric field then d may be taken to be related to charge per unit area and the case may be taken as the analogue of an infinite charged plane.

SOLUTION OF FIELD EQUATIONS II

In this case the additional equation (4*e*), can be solved on using (4*a*) and the solution gives

$$IG'G^{1/2}=f, \ (f, \text{ constant}) \qquad \ldots \quad (5f)$$

This is inconsistent with (5*c*) unless both the constants f and d vanish. This leads to either $I=0$, or $G'=0$. $I=0$ makes the field purely gravitational. $G'=0$ leads to (as can easily be seen from field equations)

$$G=\text{const}, \ H=\text{const}, \ I=\text{const}.$$

Thus the only solution in which electromagnetic field exists is one in which the symmetric elements are those corresponding to a flat space. This solution is also included in I as II is a stronger set.

CONCLUSION

If the symmetric parts of g's are interpreted as the metric and $g_{4\mu}$, as electric field then for an infinite charged plane massless charge as well as charged mass are possible according to the older field equations (1946) but in the new field equations (1950) massless charge is the only possibility if charge is to exist.

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